

Graphs with large generalized 3-connectivity *

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Abstract

Let S be a nonempty set of vertices of a connected graph G . A collection T_1, \dots, T_ℓ of trees in G is said to be internally disjoint trees connecting S if $E(T_i) \cap E(T_j) = \emptyset$ and $V(T_i) \cap V(T_j) = S$ for any pair of distinct integers i, j , where $1 \leq i, j \leq r$. For an integer k with $2 \leq k \leq n$, the generalized k -connectivity $\kappa_k(G)$ of G is the greatest positive integer r such that G contains at least r internally disjoint trees connecting S for any set S of k vertices of G . Obviously, $\kappa_2(G)$ is the connectivity of G . In this paper, sharp upper and lower bounds of $\kappa_3(G)$ are given for a connected graph G of order n , that is, $1 \leq \kappa_3(G) \leq n - 2$. Graphs of order n such that $\kappa_3(G) = n - 2$, $n - 3$ are characterized, respectively.

Keywords: connectivity, internally disjoint trees, generalized connectivity.

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1 Introduction

All graphs in this paper are undirected, finite and simple. We refer to book [1] for graph theoretical notation and terminology not described here.

The generalized connectivity of a graph G , which was introduced by Chartrand et al. in [2], is a natural and nice generalization of the concept of connectivity. A tree T is called an S -tree if $S \subseteq V(T)$, where $S \in V(G)$. A collection T_1, \dots, T_ℓ of trees in G is said to be *internally disjoint trees connecting* S if $E(T_i) \cap E(T_j) = \emptyset$ and

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$V(T_i) \cap V(T_j) = S$ for any pair of distinct integers i, j , where $1 \leq i, j \leq r$. For an integer k with $2 \leq k \leq n$, the *generalized k -connectivity* $\kappa_k(G)$ of G is the greatest positive integer r such that G contains at least r internally disjoint trees connecting S for any set S of k vertices of G . Obviously, $\kappa_2(G)$ is the connectivity of G . By convention, for a connected graph with less than k vertices, we set $\kappa_k(G) = 1$; for a disconnected graph G , we set $\kappa_k(G) = 0$.

In addition to being natural combinatorial measures, the generalized connectivity can be motivated by their interesting interpretation in practice. For example, suppose that G represents a network. If one considers to connect a pair of vertices of G , then a path is used to connect them. However, if one wants to connect a set S of vertices of G with $|S| \geq 3$, then a tree has to be used to connect them. This kind of tree with minimum order for connecting a set of vertices is usually called a Steiner tree, and popularly used in the physical design of VLSI, see [10]. Usually, one wants to consider how tough a network can be, for the connection of a set of vertices. Then, the number of totally independent ways to connect them is a measure for this purpose. The generalized k -connectivity can serve for measuring the capability of a network G to connect any k vertices in G .

There have appeared many results on the generalized connectivity, see [2, 3, 9, 4, 5, 6, 7, 8]. Chartrand et al. in [3] obtained the following result in the generalized connectivity.

Lemma 1. [3] *For every two integers n and k with $2 \leq k \leq n$,*

$$\kappa_k(K_n) = n - \lceil k/2 \rceil.$$

The following result is given by Li et al. in [7], which will be used later.

Lemma 2. [7] *For any connected graph G , $\kappa_3(G) \leq \kappa(G)$. Moreover, the upper bound is sharp.*

In Section 2, sharp upper and lower bounds of $\kappa_3(G)$ are given for a connected graph G of order n , that is, $1 \leq \kappa_3(G) \leq n - 2$. Moreover, graphs of order n such that $\kappa_3(G) = n - 2, n - 3$ are characterized, respectively.

2 Graphs with 3-connectivity $n - 2, n - 3$

For a graph G , let $V(G)$, $E(G)$ be the set of vertices, the set of edges, respectively, and $|G|$ and $\|G\|$ the order, the size of G , respectively. If S is a subset of vertices of a graph G , the subgraph of G induced by S is denoted by $G[S]$. If M is a subset

of edges of G , the subgraph of G induced by M is denoted by $G[M]$. As usual, the *union* of two graphs G and H is the graph, denoted by $G \cup H$, with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Let mH be the disjoint union of m copies of a graph H . For $U \subseteq V(G)$, we denote $G \setminus U$ the subgraph by deleting the vertices of U along with the incident edges from G . Let $d_G(v)$, simply denoted by $d(v)$, be the degree of a vertex v , and let $N_G(v)$ be the neighborhood set of v in G . A subset M of $E(G)$ is called a *matching* in G if its elements are such edges that no two of them are adjacent in G . A matching M saturates a vertex v , or v is said to be *M -saturated*, if some edge of M is incident with v ; otherwise, v is *M -unsaturated*. M is a *maximum matching* if G has no matching M' with $|M'| > |M|$.

Observation 1. *If G is a graph obtained from the complete graph K_n by deleting an edge set M and $\Delta(K_n[M]) \geq 3$, then $\kappa_3(G) \leq n - 4$.*

The observation above indicates that if $\kappa_3(G) \geq n - 3$, then each component of $K_n[M]$ must be a path or a cycle.

After the preparation above, we start to give our main results of this paper. At first, we give the bounds of $\kappa_3(G)$.

Proposition 1. *For a connected graph G of order n ($n \geq 3$), $1 \leq \kappa_3(G) \leq n - 2$. Moreover, the upper and lower bounds are sharp.*

Proof. It is easy to see that $\kappa_3(G) \leq \kappa_3(K_n)$. From this together with Lemma 1, we have $\kappa_3(G) \leq n - 2$. Since G is connected, $\kappa_3(G) \geq 1$. The result holds.

It is easy to check that the complete graph K_n attains the upper bound and the complete bipartite graph $K_{1,n-1}$ attains the lower bound. \square

Theorem 1. *For a connected graph G of order n , $\kappa_3(G) = n - 2$ if and only if $G = K_n$ or $G = K_n \setminus e$.*

Proof. Necessity If $G = K_n$, then we have $\kappa_3(G) = n - 2$ by Lemma 1. If $G = K_n \setminus e$, it follows by Proposition 1 that $\kappa_3(G) \leq n - 2$. We will show that $\kappa_3(G) \geq n - 2$. It suffices to show that for any $S \subseteq V(G)$ such that $|S| = 2$, there exist $n - 2$ internally disjoint S -trees in G .

Let $e = uv$, and $W = G \setminus \{u, v\} = \{w_1, w_2, \dots, w_{n-2}\}$. Clearly, $G[W]$ is a complete graph of order $n - 2$.

If $|\{u, v\} \cap S| = 1$ (See Figure 1 (a)), without loss of generality, let $S = \{u, w_1, w_2\}$. The trees $T_i = w_i u \cup w_i w_1 \cup w_i w_2$ together with $T_1 = u w_1 \cup w_1 w_2$, $T_2 = u w_2 \cup v w_2 \cup v w_1$ form $n - 2$ pairwise internally disjoint S -trees, where $i = 2, \dots, n - 2$.

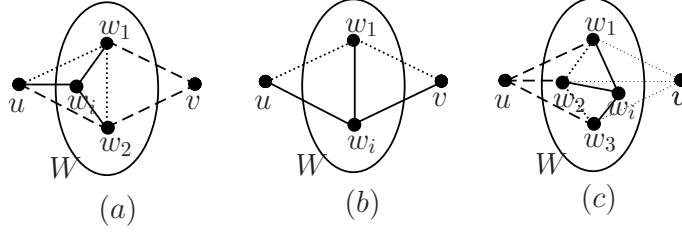


Figure 1 The edges of a tree are by the same type of lines.

If $|\{u, v\} \cap S| = 2$ (See Figure 1 (b)), without loss of generality, let $S = \{u, v, w_1\}$. The trees $T_i = w_i u \cup w_i v \cup w_i w_1$ together with $T_1 = u w_1 \cup w_1 v$ form $n - 2$ pairwise internally disjoint S -trees, where $i = 2, \dots, n - 2$.

Otherwise, suppose $S \subseteq W$ (See Figure 1 (c)). Without loss of generality, let $S = \{w_1, w_2, w_3\}$. The trees $T_i = w_i w_1 \cup w_i w_2 \cup w_i w_3$ ($i = 4, 5, \dots, n - 2$) together with $T_1 = w_2 w_1 \cup w_2 w_3$ and $T_2 = u w_1 \cup u w_2 \cup u w_3$ and $T_3 = v w_1 \cup v w_2 \cup v w_3$ form $n - 2$ pairwise internally disjoint S -trees.

From the arguments above, we conclude that $\kappa_3(K_n \setminus e) \geq n - 2$. From this together with Proposition 1, $\kappa(K_n \setminus e) = n - 2$.

Sufficiency Next we show that if $G \neq K_n, K_n \setminus e$, then $\kappa_3(G) \leq n - 3$, where G is a connected graph. Let G be the graph obtained from K_n by deleting two edges. It suffices to prove that $\kappa_3(G) \leq n - 3$. Let $G = K_n \setminus \{e_1, e_2\}$, where $e_1, e_2 \in E(K_n)$. If e_1 and e_2 has a common vertex and form a P_3 , denoted by v_1, v_2, v_3 . Thus $d_G(v) = n - 3$. So $\kappa_3(G) \leq \delta(G) \leq n - 3$. If e_1 and e_2 are independent edges. Let $e_1 = xy$ and $e_2 = vw$. Let $S = \{x, y, v\}$. We consider the internally disjoint S -trees. It is easy to see that $d_G(x) = d_G(y) = d_G(v) = n - 2$. Furthermore, each edge incident to x (each neighbor adjacent to x) in G belongs to an S -tree so that we can obtain $n - 2$ S -trees. The same is true for the vertices y and v . Let \mathcal{T} be a set of internally disjoint S -trees that contains as many S -trees as possible and $U = N_G(x) \cap N_G(y) \cap N_G(v)$. There exist at most $|U| = n - 4$ S -tree in \mathcal{T} that contain at least one vertex in U . Next we show that there exist one S -tree in $G \setminus U$. Suppose that there exist two internally disjoint S -trees in $G \setminus U$. Since $G \setminus U$ is cycle of order 4, and there exists at most one S -tree in $G \setminus U$. So $\kappa_3(G) = |\mathcal{T}| \leq n - 3$. \square

Theorem 2. Let G be a connected graph of order n ($n \geq 3$). $\kappa_3(G) = n - 3$ if and only if G is a graph obtained from the complete graph K_n by deleting an edge set M such that $K_n[M] = P_4$ or $K_n[M] = P_3 \cup P_2$ or $K_n[M] = C_3 \cup P_2$ or $K_n[M] = rP_2$ ($2 \leq r \leq \lfloor \frac{n}{2} \rfloor$).

Proof. Sufficiency. Assume that $\kappa_3(G) = n - 3$. Then $|M| \geq 2$ by Theorem 1 and

each component of $K_n[M]$ is a path or a cycle by Observation 1. We will show that the following claims hold.

Claim 1. $K_n[M]$ has at most one component of order larger than 2.

Suppose, to the contrary, that $K_n[M]$ has two components of order larger than 2, denoted by H_1 and H_2 (See Figure 2 (a)). Pick a set $S = \{x, y, z\}$ such that $x, y \in H_1$, $z \in H_2$, $d_{H_1}(y) = d_{H_2}(z) = 2$, and x is adjacent to y in H_1 . Since $d_G(y) = n - 1 - d_{H_1}(y) = n - 3$, each edge incident to y (each neighbor adjacent to y) in G belongs to an S -tree so that we can obtain $n - 3$ internally disjoint S -trees. The same is true for the vertex z . The same is true for the vertices y and v . Let \mathcal{T} be a set of internally disjoint S -trees that contains as many S -trees as possible and U be the vertex set whose elements are adjacent to both of y and z . There exist at most $|U| = n - 6$ S -trees in \mathcal{T} that contain a vertex in U .

Next we show that there exist at most 2 S -trees in $G \setminus U$ (See Figure 2 (a)). Suppose that there exist 3 internally disjoint S -trees in $G \setminus U$. Since $d_{G \setminus U}(y) = d_{G \setminus U}(z) = 3$, yz must be in an S -tree, say T_{n-5} . Then we must use one element of the edge set $E_1 = \{zx, v_2z, v_3y, v_1y\}$ if we want to reach x in T_{n-5} . Thus $d_{T_{n-5}}(y) = 2$ or $d_{T_{n-5}}(z) = 2$, which implies that there exists at most one S -tree except T_{n-5} in $G \setminus U$. So $\kappa_3(G) = |\mathcal{T}| \leq n - 4$, a contradiction.

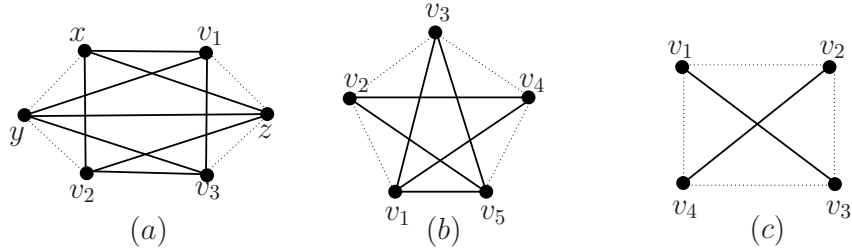


Figure 2 Graphs for Claim 1 and Claim 2(The dotted lines stand for edges in M).

Claim 2. If H is a component of $K_n[M]$ of order larger than three, then $K_n[M] = P_4$.

Suppose, to the contrary, that H is a path or a cycle of order larger than 4, or a cycle of order 4, or H is a path of order 4 and $K_n[M]$ has another component.

If H is a path or a cycle of order larger than 4, we can pick a P_5 in H . Let $P_5 = v_1, v_2, v_3, v_4, v_5$ (See Figure 2 (b)) and $S = \{v_2, v_3, v_4\}$. Since $d_H(v_2) = d_H(v_3) = d_H(v_4) = 2$, $d_G(v_2) = d_G(v_3) = d_G(v_4) = n - 3$. Furthermore, each edge incident to v_2 (each neighbor adjacent to v_2) in G belongs to an S -tree so that we can obtain $n - 3$ S -trees. The same is true for the vertices y and z . Let \mathcal{T} be a set of internally disjoint S -trees that contains as many S -trees as possible and $U = N_G(v_2) \cap N_G(v_3) \cap N_G(v_4)$. There exist at most $|U| = n - 5$ S -tree in \mathcal{T} that contain at least one vertex

in U . Next we show that there exist at most one S -tree in $G \setminus U$ (See Figure 2 (b)). Suppose that there exist two internally disjoint S -trees in $G \setminus U$. Since $d_{G \setminus U}(v_2) = d_{G \setminus U}(v_4) = 2$, v_2v_4 must be in an S -tree, say T_{n-5} . Then we must use one element of $\{v_1, v_5\}$ if we want to reach v_3 in T_{n-5} . This implies that there exists at most one S -tree except T_{n-5} in $G \setminus U$. So $\kappa_3(G) = |\mathcal{T}| \leq n - 4$, a contradiction.

If H is a cycle of order 4, let $H = v_1, v_2, v_3, v_4$ (See Figure 2 (c)), and $S = \{v_1, v_2, v_3\}$. Since $d_H(v_1) = d_H(v_2) = d_H(v_3) = 2$, $d_G(v_1) = d_G(v_2) = d_G(v_3) = n - 3$. Furthermore, each edge incident to v_1 in G belongs to an S -tree so that we can obtain $n - 3$ S -trees. The same is true for the vertices v_2 and v_3 . Let \mathcal{T} be a set of internally disjoint S -trees that contains as many S -trees as possible and $U = N_G(v_2) \cap N_G(v_3) \cap N_G(v_4)$. There exist at most $|U| = n - 4$ S -trees in \mathcal{T} that contain at least one vertex in U . It is obvious that $G \setminus U$ is disconnected, and we will show that there exists no S -tree in $G \setminus U$ (See Figure 2 (c)). So $\kappa_3(G) = |\mathcal{T}| \leq n - 4$, a contradiction. .

Otherwise, H is a path order 4 and $K_n[M]$ has another component. By Claim 1, the component must be an edge, denoted by $P_2 = u_1u_2$. Let $H = P_4 = v_1, v_2, v_3, v_4$ (See Figure 3 (a)) and $S = \{v_2, v_3, u_1\}$. Since $d_H(v_2) = d_H(v_3) = 2$, we have $d_G(v_2) = d_G(v_3) = n - 3$. Furthermore, each edge incident to v_2 (each neighbor adjacent to v_2) in G belongs to an S -tree so that we can obtain $n - 3$ S -trees. The same is true for the vertex v_3 . Let \mathcal{T} be a set of internally disjoint S -trees that contains as many S -trees as possible and U be the vertex set whose elements are adjacent to both of v_2, v_3 and u_1 . There exist at most $|U| = n - 6$ S -trees in \mathcal{T} that contain at least one vertex in U . Next we show that there exist at most two S -trees in $G \setminus U$. Suppose that there exist 3 internally disjoint S -trees in $G \setminus U$. Since $d_{G \setminus U}(v_2) = d_{G \setminus U}(v_3) = 3$, each edge incident to v_2 (each neighbor adjacent to v_2) in G belongs to an S -tree so that we can obtain 3 S -trees. The same is true for the vertex v_3 . This implies that v_2u_2 belongs to an S -trees, denoted by T_1 , and v_3u_2 belongs to an S -trees, denoted by T_2 . Clearly, $T_1 = T_2$. Otherwise, $u_2 \in T_1 \cap T_2$, which contradicts to that T_1 and T_2 are internally disjoint S -trees. Then $v_2u_2, v_3u_2 \in E(T_1)$. If we want to form T_1 , we need the vertex v_1 or v_4 . Without loss of generality, let $v_1 \in V(T_1)$. It is easy to see that there exists exactly one S -tree except T_1 in $G \setminus U$ (See Figure 3 (b)), which implies that $\kappa_3(G) \leq n - 4$. So $\kappa_3(G) = |\mathcal{T}| \leq n - 4$, a contradiction.

Claim 3. If H is a component of $K_n[M]$ of order 3, then $K_n[M] = C_3 \cup P_2$ or $K_n[M] = P_3 \cup P_2$.

By the similar arguments to the claims above, we can deduce the claim.

From the arguments above, we can conclude that G is a graph obtained from the

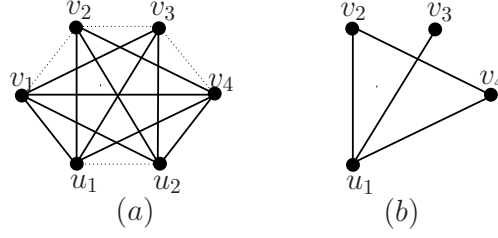


Figure 3 Graphs for Claim 2(The dotted lines stands for edges in M).

complete graph K_n by deleting an edge set M such that $K_n[M] = P_4$ or $K_n[M] = P_3 \cup P_2$ or $K_n[M] = C_3 \cup P_2$ or $K_n[M] = rP_2$ ($2 \leq r \leq \lfloor \frac{n}{2} \rfloor$).

Necessity. We show that $\kappa_3(G) \geq n-3$ if G is a graph obtained from the complete graph K_n by deleting an edge set M such that $K_n[M] = P_4$ or $K_n[M] = P_3 \cup P_2$ or $K_n[M] = C_3 \cup P_2$ or $K_n[M] = rP_2$ ($2 \leq r \leq \lfloor \frac{n}{2} \rfloor$). We consider the following cases:

Case 1. $K_n[M] = rP_2$ ($2 \leq r \leq \lfloor \frac{n}{2} \rfloor$).

In this case, M is a matching of K_n . We only need to prove that $\kappa_3(G) \geq n-3$ when M is a maximum matching of K_n . Let $S = \{x, y, z\}$. Since $|S| = 3$, S contains at most a pair of adjacent vertices under M .

If S contains a pair of adjacent vertices under M , denoted by x and y , then the trees $T_i = w_i x \cup w_i y \cup w_i z$ together with $T_1 = xy \cup yz$ form $n-3$ pairwise internally disjoint trees connecting S , where $\{w_1, w_2, \dots, w_{n-4}\} = V(G) \setminus \{x, y, z, z'\}$ such that z' is the adjacent vertex of z under M if z is M -saturated, or z' is any vertex in $V(G) \setminus \{x, y, z\}$ if z is M -unsaturated. If S contains no pair of adjacent vertices under M , then the trees $T_i = w_i x \cup w_i y \cup w_i z$ together with $T_1 = yx \cup xy' \cup y'z$ and $T_2 = yx' \cup zx' \cup zx$ and $T_3 = zy \cup yz' \cup z'x$ form $n-3$ pairwise edge-disjoint S -trees, where $\{w_1, w_2, \dots, w_{n-6}\} = V(G) \setminus \{x, y, z, x', y', z'\}$, x', y', z' are the adjacent vertices of x, y, z under M , respectively, if x, y, z are all M -saturated, or one of x', y', z' is any vertex in $V(G) \setminus \{x, y, z\}$ if the vertex is M -unsaturated.

From the arguments above, we know that $\kappa(S) \geq n-3$ for $S \subseteq V(G)$. Thus $\kappa_3(G) \geq n-3$. From this together with Theorem 1, we know $\kappa_3(G) = n-3$.

Case 2. $K_n[M] = C_3 \cup P_2$ or $K_n[M] = P_3 \cup P_2$.

If $\kappa_3(G) \geq n-3$ for $K_n[M] = C_3 \cup P_2$, then $\kappa_3(G) \geq n-3$ for $K_n[M] = P_3 \cup P_2$. So we only consider the former. Let $C_3 = v_1, v_2, v_3$ and $P_2 = u_1 u_2$, and let $S = \{x, y, z\}$ be a 3-set of G . If $S = V(C_3)$, then there exist $n-3$ pairwise internally disjoint S -trees since each vertex in S is adjacent to each vertex in $G \setminus S$. Suppose $S \neq V(C_3)$.

If $|S \cap V(C_3)| = 2$, without loss of generality, assume that $x = v_1$ and $y = v_2$. When $S \cap V(P_2) \neq \emptyset$, say $z = u_1$, the trees $T_i = w_i x \cup w_i y \cup w_i z$ together with $T_{n-4} = xz \cup yz$ and $T_{n-3} = xu_2 \cup u_2 v_3 \cup zv_3 \cup u_2 y$ form $n-3$ pairwise internally disjoint trees

connecting S , where $\{w_1, w_2, \dots, w_{n-5}\} = V(G) \setminus \{x, y, z, u_2, v_3\}$. When $S \cap V(P_2) = \emptyset$, the trees $T_i = w_i x \cup w_i y \cup w_i z$ together with $T_{n-3} = xz \cup zy$ are $n-3$ pairwise internally disjoint trees connecting S , where $\{w_1, w_2, \dots, w_4\} = V(G) \setminus \{x, y, z, v_3\}$.

If $|S \cap V(C_3)| = 1$, without loss of generality, assume $x = v_1$. When $|S \cap V(P_2)| = 2$, say $y = u_1$ and $z = u_2$, the trees $T_i = w_i x \cup w_i y \cup w_i z$ together with $T_{n-4} = xz \cup v_2 z \cup v_2 y$ and $T_{n-3} = xy \cup yv_3 \cup zv_3$ form $n-3$ pairwise internally disjoint trees connecting S , where $\{w_1, w_2, \dots, w_{n-5}\} = V(G) \setminus \{x, y, z, v_2, v_3\}$. When $|S \cap V(P_2)| = 1$, say $u_1 = y$, the trees $T_i = w_i x \cup w_i y \cup w_i z$ together with $T_{n-5} = xz \cup zy$ and $T_{n-4} = xu_2 \cup u_2 v_2 \cup v_2 y \cup v_2 z$ and $T_{n-3} = xz \cup zv_3 \cup v_3 y$ are $n-3$ pairwise internally disjoint trees connecting S , where $\{w_1, w_2, \dots, w_{n-6}\} = V(G) \setminus \{x, y, z, v_2, v_3, u_2\}$. When $|S \cap V(P_2)| = \emptyset$, the trees $T_i = w_i x \cup w_i y \cup w_i z$ together with $T_{n-4} = xz \cup zy$ and $T_{n-3} = xy \cup yv_3 \cup zv_3$ form $n-3$ pairwise internally disjoint S -trees, where $\{w_1, w_2, \dots, w_{n-5}\} = V(G) \setminus \{x, y, z, v_2, v_3\}$.

If $S \cap V(C_3) = \emptyset$, when $|S \cap V(P_2)| = 0$ or $|S \cap V(P_2)| = 2$, the trees $T_i = w_i x \cup w_i y \cup w_i z$ form $n-3$ pairwise internally disjoint S -trees, where $\{w_1, w_2, \dots, w_{n-3}\} = V(G) \setminus \{x, y, z\}$. When $|S \cap V(P_2)| = 1$, say $u_1 = x$, the trees $T_i = w_i x \cup w_i y \cup w_i z$ together with $T_{n-3} = xz \cup zy$ form $n-3$ pairwise internally disjoint S -trees, where $\{w_1, w_2, \dots, w_{n-4}\} = V(G) \setminus \{x, y, z, u_2\}$.

From the arguments above, we conclude that $\kappa(S) \geq n-3$ for $S \subseteq V(G)$. Thus $\kappa_3(G) \geq n-3$. From this together with Theorem 1, it follows that $\kappa_3(G) = n-3$.

Case 3. $K_n[M] = P_4$.

This case can be proved by an argument similar to Cases 1 and 2. \square

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